Thue's Theorems and Avoidable Patterns

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Abstract:

In 1906, Norwegian mathematician Axel Thue showed that there are arbitrarily long nonrepetitive sequences over three symbols. This was the starting point of combinatorics on words. The proof of Thue's theorem is based on a sequence of binary numbers $0, 01, 0110, 01101001, \ldots$, where the next number is the previous one followed by its negation. This sequence, first used Prouhet (1851), was also rediscovered by Morse (1921) in his work on recurrent geodesics and used by Morse and Hedlund in their paper on symbolic dynamics (1944). Use of Thue sequences was also made by Novikov and Adjan (1968) in a (negative) solution of the Burnside problem.

In this talk, we will prove the two Thue's theorems:

Theorem 0.1. *Prouhet-Thue-Morse sequence avoids patterns of type xxx.*

Theorem 0.2. *There is an infinite sequence on three letters that doesn't have repetitive blocks (avoids patterns of type xx).*

We say that a pattern is *n-avoidable* if there is an infinite sequence on n letters that avoids the pattern. Hence, xxx is 2-avoidale and xx is 3-avoidable, but not 2-avoidable. The simplest 4-avoidable pattern which is not 3-avoidable is $xytyzuzxvyxwxz$ (Baker, McNulty, Taylor (1989)). Clark (2001) found a pattern that is 5-avoidable but not 4-avoidable. Is there a 6-avoidable pattern that is not 5-avoidable?

If time permits, we will discuss several applications of Prouhet-Thue-Morse sequence and formulate some open problems.

The talk requires no specialized background.