Geometry, Combinatorics, Patterns and Fun

Niraj Khare
Carnegie Mellon University Qatar

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A few puzzles in Combinatorics

- Arranging points in a plane with a certain property (Sylvester-Gallai)

- An increasing or a decreasing sub sequence in a sequence of numbers (Erdős- Szekeress)

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Non-collinear points, special and ordinary line(s) relative to S

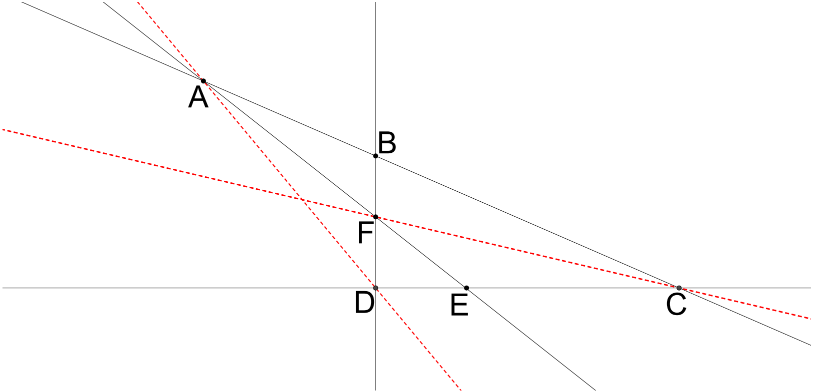

Figure: Non-collinear points, special and ordinary line(s) relative to S

Proof of Gallai-Sylvester

On the contrary assume that there is a set of finitely many points S that satisfy the hypothesis but points in S are non-collinear.

So a pair (l, a_l) exists such that l passes through two points of S , $a_l \in S$ doesn't lie on l and the distance of a_l from l is the least among all possible "line-point pairs".

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Extending to infinity

Is there a set of infinitely many non-collinear points in $\mathbb{R}^2$ such that each line passing through any two of theses points contains at least an additional (third) point from the set?

Infinite grid with integer points

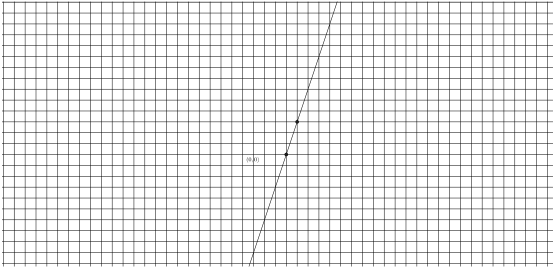

Figure: Non-collinear points satisfying Gallai-Sylvester hypothesis

Line through (i, j) and (k, l).

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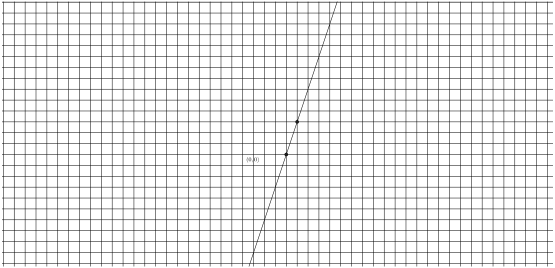

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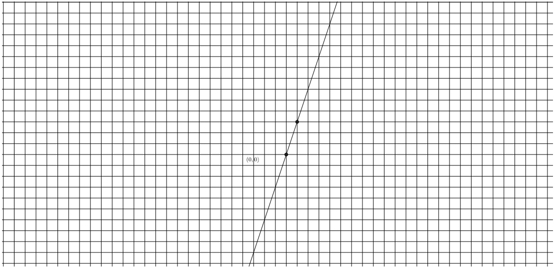

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A subsequence is an ordered collection of elements from a sequence where elements maintain the original order

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(a) i_l is an integer for all $l \in [k]$,
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For any positive integer n, among any distinct $n^2 + 1$ real numbers: either there is an increasing sub-sequence of length $n + 1$ or a decreasing sub-sequence of length $n + 1$.

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Table: largest (increasing, decreasing) sub-sequences

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Define, for each $i \in [n^2+1]$ relative to the sequence $T := (a_1, a_2, \cdots, a_{n^2+1})$ a map
$f_T : \quad [n^2+1] \to [n^2+1] \times [n^2+1]$

$$f_T(i) = (i_1, i_2)$$

where i_1 is the length of a largest increasing subsequence that ends in a_i and i_2 is the length of the largest decreasing subsequence that ends in a_i.

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Consider the sequence $(-3, 4, 1, 5, 11, 2, -2)$.

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Pigeonhole commands

Among three terms of an A.P. at least two must be of the same kind (i.e., even or odd).

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Linear translation preserves an arithmetic progression

Let $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ be two sequence of real numbers such that for each $i \in [n]$, $b_i = ka_i + c$ for some reals $k \neq 0$ and c.

Lemma (Gauss at −3)

The sequence $(a_1, a_2, \cdots, a_n)$ has a sub-sequence that is an A.P. of length m if and only if $(b_1, b_2, \cdots, b_n)$ has a sub-sequence that is an A.P. of length m.

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Extending to other fields and planes

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Ben Green, Terence Tao

Theorem (Green, Tao)

Let S be a non-collinear set of n points. If n is large (enough) then there are at least $\lfloor \frac{n}{2} \rfloor$ ordinary lines relative to S.

Gap between consecutive primes

Some primes: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$
$53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, \cdots,$
$3756801695685 \times 2^{666669} - 1$, $3756801695685 \times 2^{666669} + 1$, ...

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On number lane some primes always live close by

Theorem (Yitang Zhang)

For every positive integer n, there are distinct primes $p_1 \geq n$, $p_2 \geq n$ such that $|p_1 - p_2|$ is bounded by seventy million.

Extending Gallai-Sylvester in 3-D

Consider the following extension of Gallai-Sylvester:
Let S be a set of $n \geq 4$ points in space. If any plane that passes through three points of S contains k(n) additional point of S then S is coplanar.

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Monochromatic arithmetic progression in colorful numbers

Definition

For any positive integers n, let $[n] := \{1, 2, \cdots, n\}$. For a positive integer r, a r-coloring of $[n]$ is a function $f: \quad [n] \to [r]$.

Is there always a monochromatic arithmetic progression of any finite size, say 3, in any r-coloring of $(1, 2, \cdots, n)$ for a large enough $n >> r$?

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Van der Waerden's Theorem

Theorem (Van der Waerden)

For all positive integers k and r, there exists an integer $W(r,k)$ such that for any $n \geq W(r,k)$ and any r-coloring of $(1, 2, \cdots, n)$ there is a monochromatic arithmetic progression of size k in $(1, 2, \cdots, n)$.

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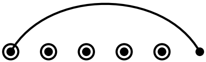
Example–patterns

- Arcs of fixed length



- crossings and nestings

- left-neighboring crossings/nestings

Patterns in partitions: Persi Daiconis's result

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matchings: partitions of $[2m]$ in which every block has size 2.
Objective: closed formula for moments of combinatorial statistics on matchings $\mathcal{M}(2m)$

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For a family of combinatorial statistics, the moments have simple closed expressions as linear combinations of double factorials $T_{2m} = (2m-1)!! = (2m-1)(2m-3)\cdots 3\cdot 1$, with constant coefficients.

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